

# The isomorphism problem for graded algebras and its application to mod- $p$ cohomology rings of small $p$ -groups

Bettina Eick and Simon King

March 17, 2015

## Abstract

The mod- $p$  cohomology ring of a non-trivial finite  $p$ -group is an infinite dimensional, finitely presented graded unital algebra over the field with  $p$  elements, with generators in positive degrees. We describe an effective algorithm to test if two such algebras are graded isomorphic. As application, we determine all graded isomorphisms between the mod- $p$  cohomology rings of all  $p$ -groups of order at most 100.

## 1 Introduction

The mod- $p$  cohomology ring  $H^*(G, \mathbb{F})$  of a non-trivial finite  $p$ -group  $G$  and the field  $\mathbb{F}$  with  $p$  elements is an infinite dimensional graded  $\mathbb{F}$ -algebra. It is an interesting and wide open question how good this algebra is as an isomorphism invariant for the underlying group  $G$ . More precisely: given two non-isomorphic  $p$ -groups  $G$  and  $H$ , under which circumstances are  $H^*(G, \mathbb{F})$  and  $H^*(H, \mathbb{F})$  isomorphic as graded  $\mathbb{F}$ -algebras?

Our aims in this paper are two-fold. First, we consider finitely presented graded unital  $\mathbb{F}$ -algebras with generators in positive degrees over a finite field  $\mathbb{F}$ ; we call such algebras ‘finitary’. We describe an effective algorithm to test if two finitary algebras are graded isomorphic. We also consider the special case of graded commutative finitary algebras and describe an improved algorithm for this case.

Secondly, we apply our algorithm to the mod- $p$  cohomology rings of the  $p$ -groups with order at most 100. The  $p$ -groups of order at most 100 are well-known, see [2] for a history on their classification. Finite presentations for their mod- $p$  cohomology rings are also available, see [4] and also [6] for a recent account. The following theorem exhibits a brief summary of our results. A complete list of the groups with graded isomorphic cohomology ring is included in Section 7 below.

**1 Theorem:** *The following tables list numbers of isomorphism types of groups of order  $p^n$  and numbers of graded isomorphism types of the associated mod- $p$  cohomology rings.*

order	$2^1$	$2^2$	$2^3$	$2^4$	$2^5$	$2^6$	$3^1$	$3^2$	$3^3$	$3^4$
# groups	1	2	5	14	51	267	1	2	5	15
# rings	1	2	5	14	48	239	1	2	5	15

There are significantly more graded isomorphisms between groups of different orders than between groups of a fixed order. In the following table we list numbers of isomorphism types of groups of order dividing  $p^n$  and numbers of graded isomorphism types of the associated mod- $p$  cohomology rings.

order	$2^1$	$2^2$	$2^3$	$2^4$	$2^5$	$2^6$	$3^1$	$3^2$	$3^3$	$3^4$
# groups	1	3	8	22	73	340	1	3	8	23
# rings	1	3	7	18	55	260	1	2	5	14

The phenomena that there are several graded isomorphisms between mod- $p$  cohomology rings for groups of different orders is known in various examples in the literature. A well-known example is given by the infinite families of cyclic groups with graded isomorphic cohomology rings. Further, there are infinite families of metacyclic groups with graded isomorphic mod- $p$  cohomology rings, see [10]. Moreover, the result in [3] implies that there are infinite families of 2-groups of fixed coclass with graded isomorphic mod-2 cohomology rings.

## 2 Preliminaries

In this preliminary section we recall some basic facts from the theory of graded algebras and we establish our notation. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and let  $\mathbb{F}$  be a field. First, recall that an  $\mathbb{F}$ -algebra  $A$  is *graded* if it can be written as a direct sum of  $\mathbb{F}$ -vectorspaces

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

and  $A_i A_j \subseteq A_{i+j}$  holds for each  $i, j \in \mathbb{N}$ . The vectorspaces  $A_0, A_1, \dots$  are the *graded components* of  $A$ . An element  $a \in A$  is *homogeneous*, if it is contained in a graded component  $A_n$  for some  $n \in \mathbb{N}$ ; in this case we denote its degree by  $|a| = n$ .

Let  $F$  be a free graded unital  $\mathbb{F}$ -algebra, let  $\varphi : F \rightarrow A$  be a surjective morphism of graded  $\mathbb{F}$ -algebras, let  $\mathcal{A}$  be a free generating set of  $F$  and let  $\mathcal{R}$  be a generating set of  $\ker(\varphi)$ . Then  $\langle \mathcal{A} \mid \mathcal{R} \rangle$  is called a *graded presentation* of  $A$ . The presentation is finite if both  $\mathcal{A}$  and  $\mathcal{R}$  are finite, and  $A$  is called *finitely presented*, if a finite presentation is given. Note that by slight abuse of notation we identify  $\mathcal{A}$  with  $\{\varphi(x) \mid x \in \mathcal{A}\}$  and say that  $\langle \mathcal{A} \mid \mathcal{R} \rangle$  is a presentation on the generating set  $\mathcal{A}$  of  $A$ .

**Definition.** Let  $\mathbb{F}$  be a finite field and let  $A$  be a finitely presented graded unital  $\mathbb{F}$ -algebra with generators in positive degrees. Then  $A$  is called a *finitary*  $\mathbb{F}$ -algebra.

**2 Lemma:** Let  $A$  be a finitary  $\mathbb{F}$ -algebra with graded components  $A_0, A_1, \dots$

- (1)  $A$  has a finite generating set consisting of homogeneous elements  $\mathcal{A} = (a_1, \dots, a_m)$ .
- (2)  $A$  has a finite presentation on the homogeneous generating set  $\mathcal{A}$ .
- (3) Let  $n \in \mathbb{N}$ . The set  $\mathcal{M}_n(\mathcal{A}) = \{a_{i_1} \cdots a_{i_j} \mid |a_{i_1}| + \dots + |a_{i_j}| = n\}$  generates  $A_n$  as vector space and thus  $A_n$  is finite dimensional.

*Proof:* (1) Each element of  $A$  can be written as a finite sum of homogenous elements. Thus each arbitrary finite generating set of  $A$  gives rise to a finite homogeneous generating set by decomposing each generator into homogeneous summands.

(2) Let  $A = \langle b_1, \dots, b_k \mid R_1, \dots, R_l \rangle$  be an arbitrary finite presentation for  $A$  and let  $a_1, \dots, a_m$  be an arbitrary finite generating set for  $A$ . Then each  $b_i$  can be written as a word in  $a_1, \dots, a_m$ , say  $b_i = w_i(a_1, \dots, a_m)$ . Similarly, each  $a_j$  can be written as word in  $b_1, \dots, b_k$ , say  $a_j = v_j(b_1, \dots, b_k)$ . It now follows that  $A \cong \langle a_1, \dots, a_m \mid R_i(w_1, \dots, w_k) \text{ for } 1 \leq i \leq l \text{ and } a_j = v_j(w_1, \dots, w_k) \text{ for } 1 \leq j \leq m \rangle$ .

(3) Is elementary. •

Let  $A$  be a graded  $\mathbb{F}$ -algebra with graded components  $A_0, A_1, \dots$ . We define

$$I(A) = \bigoplus_{n \geq 1} A_n \quad \text{and} \quad I_j(A) = \bigoplus_{n \geq j} A_n \text{ for } j \geq 1.$$

Then  $I(A)$  is called the *augmentation ideal* of  $A$ ; it is a non-unital  $\mathbb{F}$ -algebra having the series of ideals  $I(A) = I_1(A) \geq I_2(A) \geq \dots$ . Note that

$$A = I(A) \rtimes A_0.$$

Let  $I(A) \geq I(A)^2 \geq \dots$  denote the series of power ideals in  $I(A)$ . Then  $I(A)/I(A)^c$  is a nilpotent  $\mathbb{F}$ -algebra of class  $c - 1$  for each  $c \geq 1$  by construction.

**3 Lemma:** *Let  $A$  be a finitary  $\mathbb{F}$ -algebra.*

- (1) *Let  $c \in \mathbb{N}$ . Then  $I(A)^c \leq I_c(A)$ .*
- (2)  *$I(A)$  is finitely generated and residually nilpotent.*
- (3) *Let  $c \in \mathbb{N}$ . Then  $I(A)/I(A)^c$  is finite dimensional.*
- (4) *There exists  $d \in \mathbb{N}$  with  $I_{d+1}(A) \leq I(A)^2$ .*

*Proof:* (1) We use induction on  $c$ . For  $c = 1$  we note that  $I(A)^1 = I(A) \leq I(A) = I_1(A)$ . If  $I(A)^c \leq I_c(A)$ , then  $I(A)^{c+1} = I(A)I(A)^c \leq I(A)I_c(A) \leq I_{c+1}(A)$ .

(2) Let  $a_1, \dots, a_m$  be a set of homogeneous generators in positive degrees for the unital algebra  $A$ . Then  $I(A)$  is generated by  $a_1, \dots, a_m$  as non-unital algebra. Thus  $I(A)$  is finitely generated. Further,  $\bigcap_{c \geq 1} I(A)^c = \{0\}$  by (1) and thus  $I(A)$  is residually nilpotent.

(3) A nilpotent quotient of a finitely generated algebra is finite dimensional.

(4) By (2) the algebra  $I(A)$  is finitely generated and thus it has a finite generating set  $\mathcal{A}$  of homogeneous elements. Let  $d$  be the maximal degree of a generator and let  $l > d$ . Then each monomial in  $\mathcal{M}_l(\mathcal{A})$  is a product of at least 2 elements by the definition of  $d$ . Hence  $\mathcal{M}_l(\mathcal{A}) \subseteq I(A)^2$ . Thus  $I_{d+1}(A) = \langle \mathcal{M}_l(\mathcal{A}) \mid l > d \rangle \subseteq I(A)^2$ . •

### 3 Computation with finitary algebras

In this section we describe some elementary algorithms for finitary  $\mathbb{F}$ -algebras. We assume that a finitary  $\mathbb{F}$ -algebra  $A$  is given by a finite presentation  $\langle a_1, \dots, a_m \mid R_1, \dots, R_l \rangle$  on

homogeneous generators  $\mathcal{A} = (a_1, \dots, a_m)$  with positive degrees. We denote the graded components of  $A$  by  $A_0, A_1, \dots$  and we assume that the Hilbert–Poincaré series  $P_A(t) = \sum_{n \in \mathbb{N}} \dim(A_n) t^n$  is given as rational function.

It is well-known that computations with finitely presented algebraic objects is difficult in general. For example, in the case of finitely presented groups it is in general not algorithmically possible to solve the word problem (let alone the isomorphism problem). In this section we show how this and related problems can be solved in our considered case. For  $n \geq 1$  let

$$\epsilon_n : I(A) \rightarrow I(A)/I(A)^n : a \mapsto a + I(A)^n$$

the natural epimorphism on the class- $n - 1$  nilpotent quotient of  $I(A)$ . Then the image of  $\epsilon_n$  is finite dimensional by Lemma 3.

**4 Remark:** *Let  $n \in \mathbb{N}$ . Then a basis and its structure constants table for  $I(A)/I(A)^n = \text{Im}(\epsilon_n)$  can be computed with the methods of [5] together with the images of  $a_1, \dots, a_m$  in the finite-dimensional image. This computation requires an arbitrary finite presentation for  $I(A)$ . Note that the given presentation  $\langle a_1, \dots, a_m \mid R_1, \dots, R_l \rangle$  for  $A$  defines  $I(A)$  as non-unital algebra and hence a finite presentation for  $I(A)$  is given by our setup.*

### 3.1 The word problem

Suppose that a word  $w$  in the generators  $\mathcal{A}$  is given; that is,  $w = c + \sum_{i=1}^n c_i l_i$  with  $c_i \in \mathbb{F}$  and  $l_i \in \mathcal{M}_i(\mathcal{A})$ . Our aim is to decide if  $w = 0$  in  $A$ . The following lemma translates this to an easy calculation in the finite dimensional quotient  $I(A)/I(A)^{n+1}$ .

**5 Lemma:**  $w = 0$  in  $A$  if and only if  $c_0 = 0$  and  $\epsilon_{n+1}(\sum_{i=1}^n c_i l_i) = 0$ .

*Proof:* This follows from Lemma 3 (1). •

### 3.2 Bases for the graded components

Let  $n \geq 1$  and recall that  $\mathcal{M}_n(\mathcal{A})$  generates the graded component  $A_n$  of  $A$ . The following lemma shows how to reduce this generating set to a basis via a computation in the finite dimensional quotient  $I(A)/I(A)^{n+1}$ .

**6 Lemma:**  $B_n$  is a basis for  $A_n$  if and only if  $\epsilon_{n+1}(B_n)$  is a basis for  $\langle \epsilon_{n+1}(\mathcal{M}_n(\mathcal{A})) \rangle$ .

*Proof:* This follows from Lemma 3 (1). •

We note that the dimensions of the graded components can be read off readily from the Hilbert–Poincaré series  $P_A(t)$ . Define  $P_A^{(0)}(t) := P_A(t)$  and  $P_A^{(n)}(t) := (P_A^{(n-1)}(t) - \dim(A_{n-1}))/t$  for  $n > 0$ . Then  $\dim(A_n) = P_A^{(n)}(0)$  holds.

### 3.3 Detecting generating sets

Suppose that elements  $b_1, \dots, b_k$  of  $I(A)$  are given. Our aim is to decide if these elements generate  $A$  as unital algebra. The following lemma reduces this to an elementary computation in the finite dimensional quotient  $I(A)/I(A)^2$ .

**7 Lemma:**  $b_1, \dots, b_k$  generate  $A$  (as unital algebra) if and only if  $\langle \epsilon_2(b_1), \dots, \epsilon_2(b_k) \rangle = I(A)/I(A)^2$ .

*Proof:* This follows from Lemma 3 (2). •

## 4 Graded isomorphisms between finitary algebras

In this section we exhibit our solution to the graded isomorphism problem for finitary algebras. Recall that two graded  $\mathbb{F}$ -algebras  $A$  and  $B$  are *graded isomorphic* if there exists an  $\mathbb{F}$ -algebra isomorphism  $\nu : A \rightarrow B$  that respects the grading, that is, it satisfies  $\nu(A_n) = B_n$  for each  $n \in \mathbb{N}$ . We write  $A \cong B$  if  $A$  is isomorphic to  $B$  as  $\mathbb{F}$ -algebra and  $A \cong_g B$  if  $A$  is graded isomorphic to  $B$ .

For our algorithm we assume that both finitary algebras  $A$  and  $B$  are given by finite presentations on homogeneous generators of positive degrees and we assume that their Hilbert–Poincaré series  $P_A$  and  $P_B$  are available as well. We denote the graded components of  $A$  and  $B$  by  $A_n$  and  $B_n$ , respectively.

**8 Lemma:** Let  $A$  and  $B$  be two finitary  $\mathbb{F}$ -algebras. If there exists a graded isomorphism  $\varphi : A \rightarrow B$ , then

- (a)  $P_A = P_B$ , and
- (b) If  $\mathcal{A}$  is a finite homogenous generating set for  $A$ , then  $\varphi(a) \in B_{|a|}$  for each  $a \in \mathcal{A}$ .

*Proof:* (a) and (b) both follow from the fact that  $\varphi(A_n) = B_n$  for each  $n \in \mathbb{N}$ , where  $A_n$  and  $B_n$  denotes the vectorspaces in the gradings of  $A$  and  $B$ , respectively. •

Lemma 8 (b) shows that there are only finitely many possible options for graded isomorphisms  $A \rightarrow B$ , since a finite homogenous generating set for  $A$  is given and  $B_n$  is finite for each  $n \in \mathbb{N}$ .

**9 Theorem:** Let  $A$  and  $B$  be two finitary  $\mathbb{F}$ -algebras and suppose that  $P_A = P_B$ . Let  $A = \langle a_1, \dots, a_m \mid R_1, \dots, R_l \rangle$  a finite homogenous presentation for  $A$  on generators of positive degree and let  $b_1, \dots, b_m \in B$  with  $b_i \in B_{|a_i|}$  for  $1 \leq i \leq m$ . The map  $\varphi : A \rightarrow B : a_i \mapsto b_i$  extends to a graded isomorphism if and only if

- (a)  $R_j(b_1, \dots, b_m) = 0$  for  $1 \leq j \leq l$ , and
- (b)  $b_1, \dots, b_m$  generate  $B$ .

*Proof:* First suppose that  $\varphi$  extends to a graded isomorphism. Then  $0 = \varphi(0) = \varphi(R_j(a_1, \dots, a_m)) = R_j(\varphi(a_1), \dots, \varphi(a_m)) = R_j(b_1, \dots, b_m)$  and thus (a) holds. (b) is obvious.

Now suppose that (a) and (b) hold. Then (a) yields that  $\varphi$  is an algebra homomorphism. As  $b_i \in B_{|a_i|}$  for  $1 \leq i \leq m$ , it follows that  $\varphi$  respects the grading and  $\varphi(A_n) \subseteq B_n$  for  $n \in \mathbb{N}$ . (b) asserts that  $\varphi$  is surjective. Hence  $\varphi(A_n) = B_n$  for each  $n \in \mathbb{N}$ . Finally, as  $P_A = P_B$ , we obtain that  $\varphi$  is also injective and hence a graded isomorphism.  $\bullet$

Lemma 8 and Theorem 9 induce the following method to determine a graded isomorphism  $A \rightarrow B$  if it exists. Let  $A = \langle a_1, \dots, a_m \mid R_1, \dots, R_l \rangle$  be a finite homogenous presentation on generators of positive degrees and let  $d_i = |a_i|$  for  $1 \leq i \leq m$ .

### **GradedIsomorphism**( $A, B$ )

- (1) Test if  $P_A = P_B$ ; if not, then return false.
- (2) Determine bases for  $B_{d_1}, \dots, B_{d_m}$ .
- (3) For each  $(b_1, \dots, b_m) \in B_{d_1} \times \dots \times B_{d_m}$  do
  - (a) Check that  $R_j(b_1, \dots, b_m) = 0$  for  $1 \leq j \leq l$ .
  - (b) Check that  $b_1, \dots, b_m$  generate  $B$ .
  - (c) If (a) and (b) are satisfied, then return  $(b_1, \dots, b_m)$ .
- (4) Return false;

Note that bases for  $B_{d_1}, \dots, B_{d_m}$  can be determined as in Section 3.2. Each of these spaces is finite and thus the for-loop in Step (3) is a finite loop. Step (3a) can be implemented by the method in Section 3.1. Step (3b) can be performed as in Section 3.3.

**10 Remark:** Let  $w_1 = \max\{d_i \mid 1 \leq i \leq m\}$  and let  $w_2$  denote the maximal degree of a monomial in  $R_1, \dots, R_l$ . Further, let  $w = \max\{1, w_1, w_2\}$ . Then the algorithm **GradedIsomorphism** requires the computation of  $\epsilon_{w+1}$ .

If  $I(A)/I(A)^{w+1}$  and  $I(B)/I(B)^{w+1}$  are both available, then this allows further reductions in the algorithm **GradedIsomorphism**. For example, if  $A$  and  $B$  are graded isomorphic, then  $\dim(I(A)/I(A)^c) = \dim(I(B)/I(B)^c)$  for each  $c \geq 1$  and this induces an additional condition that may be checked in Step (1) of the algorithm for all available nilpotent quotients. Further, if  $a_i \in I(A)^{c_i}$  for some  $c_i \in \mathbb{N}$ , then  $\varphi(b_i) \in I(B)^{c_i} \cap B_{d_i}$ . This can be used to obtain a reduction in Step (3) of the algorithm.

## **5 The graded commutative case**

A graded  $\mathbb{F}$ -algebra  $A$  is called *graded commutative*, if for all homogeneous elements  $x, y \in A$  the equation  $x \cdot y = (-1)^{|x| \cdot |y|} y \cdot x$  holds. If  $\text{char}(\mathbb{F}) = 2$ , then a graded commutative algebra is commutative and the free graded commutative  $\mathbb{F}$ -algebra on  $m$  generators is isomorphic to the polynomial ring on  $m$  generators. If  $\text{char}(\mathbb{F}) > 2$ , then a free graded commutative  $\mathbb{F}$ -algebra is isomorphic to a tensor product of a polynomial ring and an exterior algebra.

We present graded commutative  $\mathbb{F}$ -algebras not as quotients of free graded unital  $\mathbb{F}$ -algebras (as in Section 2), but as quotients of free graded commutative  $F$ -algebras. Hence, if  $F$  is a free graded commutative  $\mathbb{F}$ -algebra, and  $\varphi : F \rightarrow A$  is a surjective morphism of graded  $\mathbb{F}$ -algebras, and  $\mathcal{A}$  is a free generating set of  $F$  and  $\mathcal{R}$  is a generating set of  $K = \ker(\varphi)$ , then we call  $\langle \mathcal{A} \mid \mathcal{R} \rangle$  a *graded commutative presentation* of  $A$ . Note that one can choose  $\mathcal{R}$  so that its elements are homogeneous.

Finitely presented graded commutative algebras are noetherian and are either commutative or are non-commutative  $G$ -algebras, for which a Gröbner basis theory is available much similar to the commutative case [9, Chapter 1.9]. With Gröbner bases, one has an alternative way to solve computational problems than by using nilpotent quotients as in Sections 3 and 4. That approach can be more effective; in particular, this is the case if the parameter  $w$  as determined in Remark 10 is large.

Let  $\langle a_1, \dots, a_m \mid R_1, \dots, R_l \rangle$  be a finite graded commutative presentation of  $A$  corresponding to  $\varphi : F \rightarrow A$  with  $K = \ker(\varphi)$ , as above. We consider a *Gröbner basis*  $\mathcal{B} = (B_1, \dots, B_k)$  for  $K$ .

- The word problem in  $A$  can be solved by polynomial reduction with respect to  $\mathcal{B}$ , and a basis of  $A_n$  is given by those elements of  $\mathcal{M}_n(A)$  that are not divisible by any of the leading monomials of  $B_1, \dots, B_k$ .
- By [11, Proposition 3.6.6 d)], the computation of Gröbner bases also allows for an effective test whether a subset of  $A$  forms a generating set of  $A$ .
- The Hilbert–Poincaré series  $P_A(t)$  can be computed as in [9, Chapter 5.2] or [11, Chapter 5], and is a rational function. More generally, if  $I \leq A$  is an ideal generated by homogeneous elements, then the quotient ring  $A/I$  is finitary graded commutative, and its Hilbert–Poincaré series  $P_{A/I}(t)$  (to which we also refer to as the “Hilbert–Poincaré series of  $I$ ”) can be computed, too.
- The nilradical of  $A$  can in principle be computed as in [11, Chapter 4.5]. There is a more efficient alternative approach is available for cohomology rings. If  $G$  is a finite group, then the nilradical of  $H^*(G, \mathbb{F}_p)$  is formed by the elements that have nilpotent restriction to all the maximal  $p$ -elementary abelian subgroups of  $G$ , by a result of Quillen (see also [3, Theorem 8.4.3]). Based on this, the nilradicals of modular cohomology rings of finite groups can be computed by intersecting the preimages of certain explicitly given ideals under morphisms (namely restrictions) of finitely presented graded commutative  $\mathbb{F}$ -algebras. The preimages can be computed as in [9, Section 1.8.10, Remark 1.8.17], and their intersection as in [9, Section 1.7.7].
- If  $I \leq A$  is an ideal generated by homogeneous elements, then its annihilator  $\text{Ann}(I) = \{x \in A \mid \forall y \in I : y \cdot x = 0\} \leq A$  can be computed [9, Section 2.8.4].

When we test in Step (3)(b) whether elements  $b_1, \dots, b_n \in B$  generate  $B$  according to [11, Proposition 3.6.6 d)], then the computation of a Gröbner basis *in elimination order* for an ideal defined in terms of  $(b_1, \dots, b_n)$  is needed. This is potentially a very expensive operation. It is thus crucial to reduce the possible choices of  $(b_1, \dots, b_n)$  in Step (3) by other methods, as described in the rest of this section. Here, elimination is used as well, but it turns out that this is feasible and reduces the computation time drastically.

## 5.1 Early detection of non-isomorphic algebras

Comparing  $P_A(t)$  and  $P_B(t)$  as in Step (1) of Algorithm **GradedIsomorphism** allows to disprove the existence of a graded isomorphism between  $A$  and  $B$  in many cases. In addition to that, we compute the nilradicals  $\text{nilrad}(A)$  and  $\text{nilrad}(B)$  of  $A$  and  $B$ , and test if  $P_{A/\text{nilrad}(A)}(t) = P_{B/\text{nilrad}(B)}(t)$ . This may detect that  $A \not\cong_g B$  even in cases where  $P_A(t) = P_B(t)$ .

## 5.2 Reducing the list of potential generator images

We now focus on possible reductions of the images  $(b_1, \dots, b_n)$  of  $(a_1, \dots, a_n)$  to be considered in Step (3) of Algorithm **GradedIsomorphism**.

Let  $(\mathcal{A}_A, \mathcal{R}_A)$  and  $(\mathcal{A}_B, \mathcal{R}_B)$  be finite graded commutative presentations of  $A$  and  $B$ . Let  $\hat{\mathcal{A}}_A = (a_{i_1}, \dots, a_{i_k})$  be a subset of  $\mathcal{A}_A$ , and let  $b_{i_1}, \dots, b_{i_k} \in B$ . Let  $I = \langle \hat{\mathcal{A}}_A \rangle \leq A$  be the ideal generated by  $\hat{\mathcal{A}}_A$ , and  $J = \langle b_{i_1}, \dots, b_{i_k} \rangle \leq B$ . We discuss here three tests that often allow to conclude that there is no graded homomorphism mapping  $a_{i_j}$  to  $b_{i_j}$  for  $j = 1, \dots, k$ . Firstly, if  $\varphi : A \rightarrow B$  is a graded isomorphism, then  $P_{A/I}(t) = P_{B/\varphi(I)}(t)$ . Hence, if the ideals  $I \leq A$  and  $J \leq B$  have different Hilbert–Poincaré series, then the map  $a_{i_j} \mapsto b_{i_j}$  can not be extended to a graded isomorphism.

Secondly, by elimination of the variables  $\mathcal{A}_A \setminus \hat{\mathcal{A}}_A$  from the relation ideal  $\mathcal{R}_A$  as in [9, Section 1.8.2], one obtains relations  $\hat{R}_{A,1}, \dots, \hat{R}_{A,l}$  that only involve elements of  $\hat{\mathcal{A}}_A$ . If  $\varphi : A \rightarrow B$  is a graded homomorphism, then  $\hat{R}_{A,c}(\varphi(a_{i_1}), \dots, \varphi(a_{i_k})) = 0$  for all  $c = 1, \dots, l$ , which can be effectively tested using a Gröbner basis of  $\langle \mathcal{R}_B \rangle$ . Hence, if  $\hat{R}_{A,c}(b_{i_1}, \dots, b_{i_k}) \neq 0$  for some  $c = 1, \dots, l$ , then the map  $a_{i_j} \mapsto b_{i_j}$  can not be extended to a graded homomorphism.

Thirdly, one can compute the *annihilators*  $\text{Ann}(I) \leq A$  and  $\text{Ann}(J) \leq B$ . If they have different Hilbert–Poincaré series, then the map  $a_{i_j} \mapsto b_{i_j}$  can not be extended to a graded isomorphism. In principle, it would also be possible to compare the radicals of the two ideals, but we found that this does not help to improve efficiency.

How do the above tests to help simplify in Step (3) of Algorithm **GradedIsomorphism**? It suffices to restrict Step (3) to those tuples  $(b_1, \dots, b_n) \in B_{d_1} \times \dots \times B_{d_n}$  that pass the above three tests for all subsets of  $(a_1, \dots, a_n)$ . In a practical implementation, one would start by using the three tests on one-element subsets, *i.e.*, one would compute all possible elements of  $B_{d_i}$  that may occur as images of  $a_i$  under any graded algebra isomorphism, for  $i = 1, \dots, n$ . This will normally leave very few possibilities, say,  $\hat{B}_{d_i} \subseteq B_{d_i}$ . Next, one would consider all possible pairs  $(b_i, b_j) \in \hat{B}_{d_i} \times \hat{B}_{d_j}$ , and use the three tests to determine all possible images of  $(a_i, a_j)$  under any graded isomorphism, for  $i, j = 1, \dots, n$ . And so on, with larger subsets of  $\mathcal{A}_A$ .

If  $A \not\cong_g B$ , the three tests will often leave no or only very few possible choices for  $(b_1, \dots, b_n)$  in Step (3) of Algorithm **GradedIsomorphism** that need to be tested in Step (3)(b). And if  $A \cong_g B$ , then often the first possible choice of  $(b_1, \dots, b_n)$  will turn out to yield a graded isomorphism by the final test in Step (3)(b).

**11 Remark:** *One should be aware that computing the nilradicals of  $A$  and  $B$ , a graded commutative presentation of  $\langle \langle \hat{A} \rangle \rangle$ , or annihilators, can generally be computationally very*



expensive. However, in all the examples that we considered, the gain of using the additional reductions in Algorithm **GradedIsomorphism** outweighs these additional costs by far.

## 6 Examples

We summarise here some examples of cohomology rings of finite groups. A minimal graded commutative presentation for each ring has been computed with the optional `pGroupCohomology` package [7] for Sage [12]. The package uses Singular [8] for the computation of Gröbner bases, annihilators and elimination in graded commutative rings. Nilradicals are computed as described in Section 7.

### 6.1 Early detection of non-isomorphy

Let  $A$  be the mod-3 cohomology ring of the extraspecial 3-group of order 27 and exponent 3, which is group number 3 of order 27 in the small groups library [1]. Let  $B$  be the mod-3 cohomology ring of the Sylow 3-subgroup of  $U_3(8)$ , which is group number 9 of order 81 in the small groups library. Each of these algebras has a minimal graded commutative presentations with generators in degrees 1, 1, 2, 2, 2, 2, 3, 3, and 6.

The Hilbert-Poincaré series of the two algebras, respectively of their nilradicals, coincide. The power series expansion of the Hilbert-Poincaré series is

$$P_A(t) = P_B(t) = 1 + 2t + 4t^2 + 6t^3 + 7t^4 + 8t^5 + 9t^6 + 10t^7 + 12t^8 + \dots$$

Thus,  $B_1$ ,  $B_2$ ,  $B_3$  and  $B_6$  contain  $3^2 - 1 = 8$ ,  $3^4 - 1 = 80$ ,  $3^6 - 1 = 728$  and 19682 non-zero elements, respectively. Hence, without the reductions from Section 5.2, one would need to consider  $8^2 \cdot 80^4 \cdot 728^2 \cdot 19682 > 10^{19}$  possible images for the generators of  $A$ .

However, it turns out that there is one degree-2 generator  $a \in A$  so that  $P_{A/\langle a \rangle}(t)$  is different from  $P_{B/\langle b \rangle}(t)$ , for each of the 80 non-zero elements of  $B_2$ . Hence we can readily detect that  $A$  and  $B$  are not graded isomorphic.

### 6.2 A more difficult to detect pair of non-isomorphic algebras

Let  $A$  be the mod-2 cohomology ring of group number 27 of order 32 in the small groups library, and let  $B$  be the mod-2 cohomology ring of group number 128 of order 64. They both have minimal graded commutative presentations formed by three generators in degree 1 and three generators in degree 2, and four relations.

The Hilbert-Poincaré series of the algebras, respectively of their nilradicals, coincide. The power series expansion of the Hilbert-Poincaré series is

$$P_A(t) = 1 + 3t + 7t^2 + 13t^3 + 22t^4 + 34t^5 + 50t^6 + \dots$$

Thus,  $B_1$  contains  $2^3 - 1 = 7$  and  $B_2$  contains  $2^7 - 1 = 127$  non-zero elements. Hence, without the reductions from Section 5.2, one would need to consider  $7^3 \cdot 127^3 > 7 \cdot 10^8$  possible images for the generators of  $A$ .

In contrast to the previous example, the methods from Section 5.2 applied to one-element subsets of the generating set of  $A$  are not strong enough to prove  $A \not\cong_g B$ . However, when

applied to the triple of degree-1 generators, the tests only leave 6 candidates for the images of the triple under isomorphism. Applied to the three degree-1 and two of the degree-2 generators, still as many as 4608 different isomorphic images seem possible. And thus one needs to combine each of them with the 111 potential isomorphic images of the remaining degree-2 generator. In all but 176 cases, the mapping of generators does not extend to a homomorphism, and in the remaining 176 cases the homomorphism is not surjective. Hence, the two algebras are not graded isomorphic.

## 7 Application to cohomology rings

Let  $p$  be a prime, let  $G$  be a finite  $p$ -group and let  $\mathbb{F}$  be the field with  $p$  elements. Then the mod- $p$  cohomology ring  $H^*(G, \mathbb{F})$  is a graded  $\mathbb{F}$ -algebra defined by

$$H^*(G, \mathbb{F}) = \bigoplus_{n \in \mathbb{N}} H^n(G, \mathbb{F}).$$

By the theorem of Evens–Venkov (see also [3, Theorem 6.5.1]), modular cohomology rings of finite groups are finitely presentable graded-commutative algebras. Each graded component  $H^n(G, \mathbb{F})$  is a finite dimensional vectorspace over  $\mathbb{F}$  and  $H^0(G, \mathbb{F}) \cong \mathbb{F}$ . Hence  $H^*(G, \mathbb{F})$  is a finitary  $\mathbb{F}$ -algebra. The methods of [6] determine a *minimal* presentation of  $H^*(G, \mathbb{F})$  and these allow to apply the methods described in the first part of this paper.

In the following sections we exhibit the graded isomorphisms among the mod- $p$  cohomology rings of the  $p$ -groups of order at most 100. As a preliminary step we observe that the rank of the underlying  $p$ -group is an isomorphism invariant for the cohomology ring. Recall that the rank of a finite  $p$ -group  $G$  is the rank of the finite elementary abelian quotient  $G/[G, G]G^p = G/\phi(G)$  of  $G$  or, equivalently, the minimal generator number of  $G$ .

In the following we consider the  $p$ -groups of order at most 100 by their generator number. The groups with 1 generator are the cyclic groups; it is well-known that the cyclic of order  $p^n$  have graded isomorphic mod- $p$ -cohomology rings (with the exception of the cyclic group of order 2). We thus omit this case in our list below. It then remains to consider the groups of order dividing  $2^6$ ,  $3^4$ ,  $5^2$  and  $7^2$ . The cases  $5^2$  and  $7^2$  are again well-understood and hence we focus on  $2^6$  and  $3^4$  in the following exposition.

### 7.1 2-groups

We give here a complete and irredundant list of all graded isomorphic mod-2 cohomology rings  $H^*(G, \mathbb{F})$  for the groups  $G$  of order dividing 64. We identify a group  $G$  by its id `[order, number]` in the SmallGroups library, see [1].

Each of the following lists of groups satisfies that the mod-2 cohomology rings of the considered groups are pairwise graded isomorphic, and mod-2 cohomology rings of groups from different lists are not graded isomorphic. If a group of order dividing 64 does not appear in any of the lists, then the graded isomorphism type of its mod-2 cohomology ring is unique among all groups of order dividing 64.

Additionally to the ids of the groups in the list, we include the rank of the groups and in many cases also a structure description. For the latter, we denote with  $C_k, D_k, Q_k, SD_k$

the cyclic, dihedral, quaternion and semidihedral groups of order  $k$ , respectively. The symbols  $\times$ ,  $:$  and  $.$  describe a direct product, a split extension and an arbitrary extension, respectively.

If one of the groups in one of the following lists is metacyclic, then all groups are metacyclic and we include this information as well. We note that our result differ in one case from the theoretical description of the mod- $p$  cohomology rings of metacyclic groups in [10]: our results imply that the mod-2 cohomology rings of the metacyclic groups [32, 15] and [64, 49] are graded isomorphic to each other, but they are not graded isomorphic to [64, 45] as Theorem E(2) of [10] suggests. This is based on the fact that the presentations of the mod- $p$  cohomology rings of [32, 15] and [64, 49] as given in [6] and also in [4, Appendix] are not compatible with that in [10]; for example, the presentations obtained in [6] and in [4, Appendix] imply that the underlying cohomology rings have a non-nilpotent element in degree 3 in contradiction to Theorem E(2) of [10].

- groups [4, 1], [8, 1], [16, 1], [32, 1], [64, 1]  
rank 1 and cyclic
- groups [8, 2], [16, 5], [32, 16], [64, 50]  
rank 2, metacyclic, and structure  $C_{2^n} \times C_2$  ( $n > 1$ )
- groups [8, 3], [16, 7], [32, 18], [64, 52]  
rank 2, metacyclic, and structure  $D_{2^n}$  ( $n > 2$ )
- groups [16, 2], [32, 3], [32, 4], [64, 2], [64, 3], [64, 26], [64, 27]  
rank 2, metacyclic, and structure  $C_{2^n} : C_{2^m}$  ( $n, m > 1$ )
- groups [16, 3], [32, 9], [64, 38]  
rank 2, metacyclic, and structure  $(C_{2^n} \times C_2) : C_2$  ( $n > 1$ )
- groups [16, 4], [32, 12], [32, 13], [32, 14], [64, 15], [64, 16], [64, 44], [64, 47], [64, 48]  
rank 2, metacyclic, and structure  $C_{2^m} : C_{2^n}$  ( $m, n > 1$ )
- groups [16, 6], [32, 17], [64, 51]  
rank 2, metacyclic, and structure  $C_{2^n} : C_2$  ( $n > 2$ )
- groups [16, 8], [32, 19], [64, 53]  
rank 2, metacyclic, and structure  $SD_{2^n}$  ( $n > 3$ )
- groups [16, 9], [32, 20], [64, 54]  
rank 2, metacyclic, and structure  $Q_{2^n}$  ( $n > 3$ )
- groups [16, 10], [32, 36], [64, 183]  
rank 3, and structure  $C_{2^n} \times C_2 \times C_2$  ( $n > 1$ )
- groups [16, 11], [32, 39], [64, 186]  
rank 3, and structure  $C_2 \times D_{2^n}$  ( $n > 2$ )
- groups [32, 2], [64, 17], [64, 21]  
rank 2, and structure  $(C_{2^n} \times C_2) : C_4$  ( $n > 1$ )
- groups [32, 5], [64, 6], [64, 29]  
rank 2, and structure  $(C_{2^n} \times C_{2^m}) : C_2$  ( $n > m$ )
- groups [32, 10], [64, 39]  
rank 2, and structure  $Q_{2^n} : C_4$  ( $n > 2$ )
- groups [32, 15], [64, 49]

- rank 2, metacyclic, and structure  $C_4.D_{2^n}$  ( $n > 2$ )
- groups [32, 21], [64, 83], [64, 84]  
rank 3, and structure  $(C_{2^n} : C_4) \times C_2$  ( $n > 1$ )
- groups [32, 22], [64, 95]  
rank 3, and structure  $((C_{2^n} : C_2) : C_2) \times C_2$  ( $n > 1$ )
- groups [32, 23], [64, 103], [64, 106], [64, 107]  
rank 3, and structure  $(C_{2^m} : C_{2^n}) \times C_2$  ( $m, n > 1$ )
- groups [32, 25], [64, 115], [64, 118], [64, 123]  
rank 3
- groups [32, 26], [64, 126]  
rank 3, and structure  $C_{2^n} : Q_8$  ( $n > 1$ )
- groups [32, 28], [64, 140], [64, 147]  
rank 3
- groups [32, 29], [64, 155], [64, 157]  
rank 3
- groups [32, 31], [64, 167]  
rank 3, and structure  $(C_{2^n} \times C_4) : C_2$  ( $n > 1$ )
- groups [32, 34], [64, 174]  
rank 3, and structure  $(C_{2^n} \times C_4) : C_2$  ( $n > 1$ )
- groups [32, 35], [64, 181]  
rank 3, and structure  $C_{2^n} : Q_8$  ( $n > 1$ )
- groups [32, 37], [64, 184]  
rank 3, and structure  $(C_{2^n} : C_2) \times C_2$  ( $n > 2$ )
- groups [32, 38], [64, 185]  
rank 3, and structure  $(C_{2^n} \times C_2) : C_2$  ( $n > 2$ )
- groups [32, 40], [64, 187]  
rank 3, and structure  $C_2 \times SD_{2^n}$  ( $n > 3$ )
- groups [32, 41], [64, 188]  
rank 3, and structure  $C_2 \times Q_{2^n}$  ( $n > 3$ )
- groups [32, 42], [64, 189]  
rank 3, and structure  $(C_{2^n} : C_2) \times C_2$  ( $n > 2$ )
- groups [32, 43], [64, 190]  
rank 3, and structure  $(C_2 \times D_{2^n}) : C_2$  ( $n > 2$ )
- groups [32, 44], [64, 191]  
rank 3, and structure  $(C_2 \times Q_{2^n}) : C_2$  ( $n > 2$ )
- groups [32, 45], [64, 246]  
rank 4, and structure  $C_{2^n} \times C_2 \times C_2 \times C_2$  ( $n > 1$ )
- groups [32, 46], [64, 250]  
rank 4, and structure  $C_2 \times C_2 \times D_{2^n}$  ( $n > 2$ )
- groups [64, 13], [64, 14]  
rank 2
- groups [64, 31], [64, 40]

- rank 2, and structure  $(C_{16} \times C_2) : C_2$
- groups [64, 112], [64, 113]  
rank 3, and structure  $(C_4 : C_8) : C_2$
- groups [64, 119], [64, 121]  
rank 3
- groups [64, 120], [64, 122]  
rank 3, and structure  $Q_{16} : C_4$
- groups [64, 124], [64, 125]  
rank 3
- groups [64, 142], [64, 148]  
rank 3
- groups [64, 144], [64, 146]  
rank 3
- groups [64, 156], [64, 158]  
rank 3, and structure  $Q_8 : Q_8$
- groups [64, 161], [64, 162]  
rank 3, and structure  $(C_2 \times (C_4 : C_4)) : C_2$
- groups [64, 164], [64, 165]  
rank 3, and structure  $(Q_8 : C_4) : C_2$
- groups [64, 173], [64, 176]  
rank 3, and structure  $(C_8 \times C_4) : C_2$

## 7.2 3-groups

- groups [3, 1], [9, 1], [27, 1], [81, 1]  
rank 1 and cyclic
- groups [9, 2], [27, 2], [81, 2], [81, 4], [81, 5]  
rank 2, metacyclic and structure  $C_{3^n} : C_{3^m}$
- groups [27, 4], [81, 6]  
rank 2, and structure  $C_{3^n} : C_3$
- groups [27, 5], [81, 11]  
rank 3, and structure  $C_{3^n} : C_3^2$

## References

- [1] H. U. Besche, B. Eick, and E. O’Brien. *SmallGroups - a library of groups of small order*, 2005. A refereed GAP 4 package, see [13].
- [2] H. U. Besche, B. Eick, and E. A. O’Brien. A millenium project: constructing small groups. *Internat. J. Algebra Comput.*, 12:623 – 644, 2002.
- [3] J. F. Carlson. Coclass and cohomology. *J. Pure Appl. Algebra*, 200(3):251–266, 2005.

- [4] J. F. Carlson, L. Townsley, L. Valeri-Elizondo, and M. Zhang. *Cohomology rings of finite groups*, volume 3 of *Algebras and Applications*. Kluwer Academic Publishers, Dordrecht, 2003. With an appendix: Calculations of cohomology rings of groups of order dividing 64 by Carlson, Valeri-Elizondo and Zhang.
- [5] B. Eick. Computing nilpotent quotients of associative algebras and algebras satisfying a polynomial identity. *Internat. J. Algebra Comput.*, 21(8):1339–1355, 2011.
- [6] D. J. Green and S. A. King. The computation of the cohomology rings of all groups of order 128. *J. Algebra*, 325:352–363, 2011.
- [7] D. J. Green and S. A. King. *p*-Group Cohomology Package (Version 2.1.4). Peer reviewed optional package for Sage [12]. Available from <http://sage.math.washington.edu/home/SimonKing/Cohomology>.
- [8] G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR — *A computer algebra system for polynomial computations*, 2009. Available from <http://www.singular.uni-kl.de>.
- [9] G. M. Greuel and G. Pfister. *A Singular introduction to commutative algebra*. Extended edition. Springer, 2008.
- [10] J. Huebschmann. The mod- $p$  cohomology rings of metacyclic groups. *J. Pure Appl. Algebra*, 60(1):53–103, 1989.
- [11] M. Kreuzer and L. Robbiano. *Computational commutative algebra. 2*. Springer-Verlag, Berlin, 2005.
- [12] W. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2009. See <http://www.sagemath.org>.
- [13] The GAP Group. *GAP – Groups, Algorithms and Programming, Version 4.4*. Available from <http://www.gap-system.org>, 2005.